NEAR FIELD OF INTERNAL GRAVITY WAVES EXCITED BY A SOURCE IN A MOVING STRATIFIED LIQUID

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We consider the excitation of internal gravity waves by a source in a moving stratified liquid with an arbitrary Väisälä–Brunt frequency distribution $N^2(z)$. The field of the internal waves at large distances from the source is a sum of modes which propagate independently of one another [1]. The asymptotic forms of individual modes have been studied in [1-3] for different functions $N^2(z)$. In the near zone the modes cannot be separated and it is necessary to sum a large number of modes, each of which can have an essential singularity, such as a logarithmic singularity [4]. Therefore the asymptotic form of the field of internal waves near the source is an important problem.

In the present paper we consider the field of internal waves near the source and we construct the asymptotic solutions for both the vertical and horizontal velocity components of the wave. We discuss the results of numerical calculations, which show that the agreement between the asymptotic and exact solutions is good for distances comparable to the thickness of the layer of liquid in which the oscillations propagate.

A stratified liquid flows with velocity V in a layer $0 < z < H$. The vertical velocity w of an internal wave produced by a source switched on at time $t = 0$ and located in the flowing liquid, satisfies the following equation (in the Boussinesq approximation):

$$(\partial^2/\partial t^2)\Delta_3 w + N^2(z)\Delta_2 w = Q\theta(t)\,\delta''_{tt}(x+Vt)\,\delta(y)\,\delta'_{z_0}(z-z_0), \tag{1}$$

where Q is the intensity of the source; $\Delta_2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$; $\Delta_3 = \Delta_2 + \partial^2/\partial z^2$; z_0 is the depth of the source; $\theta(t) = 0$, $t < 0$; $\theta(t) = 1$, $t > 0$. In the approximation of rigid walls ($w = 0$ at $z = 0$, H) the solution of (1) in the limit $t \to \infty$ has the form [5] $w = \sum_n w_n$, $w_n = I_0$ $(\xi < 0)$, $w_n = I_0 + I_- + I_+$ $(\xi > 0)$, $\xi = x + Vt$, where

$$I_\pm = \frac{Q}{4\pi i}\int_{-\infty}^{\infty} \exp(\mp i\mu_n(\nu)\xi - i\nu y)\,A_n(\nu, z, z_0)\,d\nu; \tag{2}$$

$$I_0 = \frac{Q}{4\pi}\int_{-\infty}^{\infty} \exp(-\lambda_n(\nu)|\xi| - i\nu y)\,B_n(\nu, z, z_0)\,d\nu, \tag{3}$$

$$A_n(\nu, z, z_0) = \frac{V^2\mu_n^3(\nu)}{\mu_n^2(\nu)+\nu^2}\left(\frac{\mu_n(\nu)\mu_n'(\nu)}{\nu}+1\right)\psi_n(z,\nu)\frac{\partial\psi_n(z_0,\nu)}{\partial z_0},$$

$$B_n(\nu, z, z_0) = \frac{V^2\lambda_n^3(\nu)}{\lambda_n^2(\nu)-\nu^2}\left(\frac{\lambda_n(\nu)\lambda_n'(\nu)}{\nu}-1\right)\varphi_n(z,\nu)\frac{\partial\varphi_n(z_0,\nu)}{\partial z_0}.$$

Here the eigenfunctions $\psi_n(z, \nu)$ and eigenvalues $\mu_n(\nu)$ are solutions of the eigenvalue problem

$$\frac{\partial^2\psi_n(z,\nu)}{\partial z^2} + [\mu_n^2(\nu)+\nu^2]\left[\frac{N^2(z)}{V^2\mu_n^2(\nu)}-1\right]\psi_n(z,\nu) = 0,$$
$$\psi_n = 0,\ z = 0, H,$$

and the eigenfunctions $\varphi_n(z, \nu)$ and eigenvalues $\lambda_n(\lambda)$ are the solutions of the eigenvalue problem

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$$\frac{\partial^2\varphi_n(z,\nu)}{\partial z^2}+\left[\lambda_n^2(\nu)-\nu^2\right]\left[\frac{N^2(z)}{V^2\lambda_n^2(\nu)}+1\right]\varphi_n(z,\nu)=0, \tag{4}$$
$$\varphi_n=0,\ z=0,\ H.$$

In the limit $\xi \to 0$ the term I_0 makes the dominant contribution, since $I_- + I_+ = 0$ when $\xi = 0$. Hence, we will be interested in the asymptotic form of I_0 when $y, \xi \to 0$. The asymptotic form of the integral (3) when $y, \xi \to 0$ is determined by the behavior of the dispersion curves $\lambda_n(\nu)$, $\nu \to \infty$. Expansions for these curves were constructed in [5] for $\nu \to \infty$ and fixed n. However, they are nonuniform in n when ν is large. In contrast to [5], uniform expansions of the dispersion curves $\lambda_n(\nu)$ and the eigenfunctions $\varphi_n(z, \nu)$, are found in the form:

$$\lambda_n^2(\nu)=\sigma_n^2+\beta_n/\sigma_n^2+o\left(\sigma_n^{-2}\right)\equiv\alpha_n^2+o\left(\sigma_n^{-2}\right); \tag{5}$$
$$\varphi_n(z,\nu)=\varphi_n^0+\varphi_n^1/\sigma_n^2+o\left(\sigma_n^{-2}\right),\ \sigma_n^2=\nu^2+\pi^2n^2/H^2. \tag{6}$$

Substituting (5) and (6) into (4) and equating terms with corresponding powers of $\sigma_n{}^2$, we obtain $\varphi_n^0(z)=\sin(\pi nz/H)$ in the first approximation. Equating terms with $\sigma_n{}^{-2}$, we obtain for the second approximation $\varphi_n^1(z,\nu)$

$$\frac{\partial^2\varphi_n^1}{\partial z^2}+\frac{\pi^2n^2}{H^2}\varphi_n^1=\left[-\left(\alpha_n^2-\sigma_n^2\right)-\left(\alpha_n^2-\nu^2\right)\frac{N^2(z)}{V^2\sigma_n^2}\right]\varphi_n^0, \tag{7}$$
$$\varphi_n^1=0,\quad z=0,\ H.$$

The problem (7) is solved by the method of variation of parameters

$$\varphi_n^1(z,\nu)=S_n(z,\nu)\sin\left(\frac{\pi nz}{H}\right)+C_n(z,\nu)\cos\left(\frac{\pi nz}{H}\right).$$

To determine the functions $S_n(z, \nu)$, $C_n(z, \nu)$ we write down a linear system of equations for $\partial S_n/\partial z$, $\partial C_n/\partial z$:

$$\frac{\partial S_n}{\partial z}\sin\left(\frac{\pi nz}{H}\right)+\frac{\partial C_n}{\partial z}\cos\left(\frac{\pi nz}{H}\right)=0,\quad \frac{\partial S_n}{\partial z}\cos\left(\frac{\pi nz}{H}\right)-\frac{\partial C_n}{\partial z}\sin\left(\frac{\pi nz}{H}\right)=R_n(z,\nu),$$
$$R_n(z,\nu)=\frac{H}{\pi n}\left[-\left(\alpha_n^2-\sigma_n^2\right)-\left(\alpha_n^2-\nu^2\right)\frac{N^2(z)}{V^2\sigma_n^2}\right]\sin\left(\frac{\pi nz}{H}\right).$$

Hence

$$\frac{\partial S_n}{\partial z}=R_n(z,\nu)\cos\left(\frac{\pi nz}{H}\right),\quad \frac{\partial C_n}{\partial z}=-R_n(z,\nu)\sin\left(\frac{\pi nz}{H}\right).$$

Then to within the arbitrary constants c_1 and c_2 the eigenfunction $\varphi_n^1(z, \nu)$ has the form

$$\varphi_n^1(z,\nu)=\left[\int_0^z R_n\cos\left(\frac{\pi n\tau}{H}\right)d\tau+c_1\right]\sin\left(\frac{\pi nz}{H}\right)-\left[\int_0^z R_n\sin\left(\frac{\pi n\tau}{H}\right)d\tau+c_2\right]\cos\left(\frac{\pi nz}{H}\right).$$

Using the boundary conditions, we find that $c_2 = 0$ and we obtain a condition for $\alpha_n{}^2$:

$$\int_0^H\left[\alpha_n^2-\sigma_n^2-\left(\alpha_n^2-\nu^2\right)\frac{N^2(z)}{V^2\sigma_n^2}\right]\sin^2\left(\frac{\pi nz}{H}\right)dz=0.$$

Then

$$\lambda_n^2(\nu)=\alpha_n^2+o\left(\sigma_n^{-2}\right)=\sigma_n^2+\frac{2\pi^2n^2}{H^3V^2\sigma_n^2}\int_0^H N^2(z)\sin^2\left(\frac{\pi nz}{H}\right)dz+o\left(\sigma_n^{-2}\right). \tag{8}$$

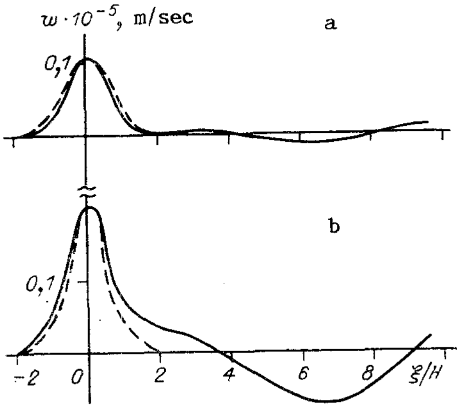

Fig. 1

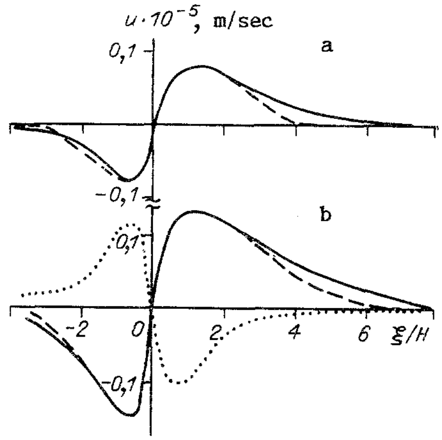

Fig. 2

The expansion (8) is uniform in n in the limit $\nu \to \infty$. According to (8), the asymptotic form of mode w_n in the limit y, $\xi \to 0$ is

$$w_n \approx I_0 \approx \frac{nQ}{2H^2} \int\limits_{-\infty}^{\infty} \exp(-\sigma_n|\xi| - i\nu y) \frac{\sin\left(\frac{\pi n z}{H}\right)\cos\left(\frac{\pi n z_0}{H}\right)}{\sigma_n} d\nu = \tag{9}$$
$$= \frac{nQ}{H^2} K_0\left(\frac{\pi n \rho}{H}\right)\sin\left(\frac{\pi n z}{H}\right)\cos\left(\frac{\pi n z_0}{H}\right) \equiv w_n^0, \quad \rho^2 = y^2 + \xi^2.$$

Here $K_0(x)$ is the MacDonald function of zero order. Hence the asymptotic form of an individual mode of the vertical velocity in the limit y, $\xi \to 0$ is given by (9). Each mode represented in this form has a logarithmic singularity when $\rho \to 0$, $z \neq z_0$, which was pointed out in [4]. The series $\sum\limits_n w_n^0$ can be summed [6]. We then obtain the total field, which is regular when $\rho \to 0$, $z \neq z_0$:

$$w \approx \sum_n w_n^0 = \frac{Q}{4\pi}\left\{\frac{z_-}{(\rho^2+z_-^2)^{3/2}} + \frac{z_+}{(\rho^2+z_+^2)^{3/2}} - \sum_{m=1}^{\infty}\left[\frac{2mH - z_-}{(\rho^2+(2mH-z_-)^2)^{3/2}} - \right.\right. \tag{10}$$
$$\left.\left. - \frac{2mH + z_-}{(\rho^2+(2mH+z_-)^2)^{3/2}} + \frac{2mH - z_+}{(\rho^2+(2mH-z_+)^2)^{3/2}} - \frac{2mH + z_+}{(\rho^2+(2mH+z_+)^2)^{3/2}}\right]\right\},$$

where $z_- = z - z_0$, $z_+ = z + z_0$.

In the numerical calculations of the vertical velocity (Fig. 1) we used the values of Q and $N^2(z)$ of [5]. The remaining parameters were taken to be: V = 6 m/sec, H = 600 m, z = 200 m, z_0 = 100 m, y = 100 m. In Fig. 1a we show the numerical results for the first mode of the vertical velocity calculated according to (2) and (3) (solid curve) and according to (9) (dashed curve). The sum of modes is shown in Fig. 1b, calculated according to (2) and (3) (solid curve) and according to (10) (dashed curve), respectively.

It is well known that the horizontal and vertical velocity components of internal waves are coupled by the equations [7]

$$\Delta_2 u + \partial^2 w/\partial x \partial z = 0, \ \Delta_2 v + \partial^2 w/\partial y \partial z = 0 \tag{11}$$

(u is the component of the velocity along the x axis and v is the component along y). We will consider the component u, since all of the calculations for v are similar. Using (11), it is not difficult to write down an expression for the horizontal velocity u:

$$u = \sum_n u_n, \ u_n = J_0 + J_- + J_+ \ (\xi > 0), \ u_n = J_0 \ (\xi < 0),$$

where

$$J_{\pm} = \frac{Q}{4\pi} \int\limits_{-\infty}^{\infty} \exp(\mp i\mu_n(\nu)\,\xi - i\nu y) \frac{\mu_n(\nu)}{\mu_n^2(\nu) + \nu^2} \frac{\partial A_n(\nu, z, z_0)}{\partial z} d\nu; \tag{12}$$

$$J_0 = \frac{Q\,\mathrm{sgn}(\xi)}{4\pi} \int\limits_{-\infty}^{\infty} \exp(-\lambda_n(\nu)|\xi| - i\nu y) \frac{\lambda_n(\nu)}{\lambda_n^2(\nu) - \nu^2} \frac{\partial B_n(\nu, z, z_0)}{\partial z} d\nu. \tag{13}$$

Using the expansion (8), we write down the asymptotic form of an individual mode in analogy with (9):

$$u_n \approx J_0 \approx \frac{Q\,\mathrm{sgn}(\xi)}{2\pi H} \int\limits_{-\infty}^{\infty} \exp(-\sigma_n|\xi| - i\nu y) \cos\left(\frac{\pi n z}{H}\right) \cos\left(\frac{\pi n z_0}{H}\right) d\nu = \\ = \frac{Qn\xi}{H^2\rho} K_1\left(\frac{\pi n \rho}{H}\right) \cos\left(\frac{\pi n z}{H}\right) \cos\left(\frac{\pi n z_0}{H}\right) \equiv u_n^0 \tag{14}$$

[$K_1(x)$ is the MacDonald function of first order]. The total field of the horizontal velocity u in this case is [6]

$$u \approx \sum_n u_n^0 = \frac{Q\xi}{2\pi H}\Big\{-\rho^{-2} + \frac{H}{2}\left[(\rho^2 + z_-^2)^{-3/2} + (\rho^2 + z_+^2)^{-3/2}\right] + \\ + \frac{H}{2} \sum_{m=1}^{\infty} [(\rho^2 + (2mH - z_-)^2)^{-3/2} + (\rho^2 + (2mH + z_-)^2)^{-3/2} + \\ + (\rho^2 + (2mH - z_+)^2)^{-3/2} + (\rho^2 + (2mH + z_+)^2)^{-3/2}]\Big\}. \tag{15}$$

We see from (15) that u has a singularity for y, $\xi \to 0$, and $z \neq z_0$, which is equal to the derivative with respect to ξ of the fundamental solution of Laplace's equation in two dimensions $\Delta_2 u = 0$ (for the velocity component v the singularity is the derivative with respect to y of the fundamental solution). Therefore to find an expression for u satisfying (11) and regular when y, $\xi \to 0$, and $z \neq z_0$, the first term must be excluded from the series (15). Then the new series will describe u when y, $\xi \to 0$.

The velocity u was calculated numerically (see Fig. 2) for the same values of the parameters. The first mode of u is shown in Fig. 2a calculated according to (12) and (13) (solid curve) and according to (14) (dashed curve). In Fig. 2b we show the sum of modes using (12) and (13) with the singularity excluded (solid curve) and using (15), also with the singularity excluded (dashed curve). The dotted curve shows the singularity of the horizontal velocity u. It follows from Fig. 2 that the horizontal velocity is dominated by the singularity when y, $\xi \to 0$.

Our results show that the explicit asymptotic expressions obtained here for the vertical and horizontal velocity components of internal waves can be used to effectively calculate the field of the internal wave at distances comparable to the thickness of the liquid layer, without running into complicated numerical calculations.

LITERATURE CITED

1. V. A. Borovikov, Yu. V. Vladimirov, and M. Ya. Kel'bert, "Field of internal gravity waves excited by localized sources," Izv. Akad. Nauk SSSR, FAO, 20, No. 6 (1984).
2. V. F. Sannikov, "Far field of steady waves created by localized sources in a moving stratified liquid," Prikl. Mat. Mekh., 50, No. 6 (1986).
3. V. A. Borovikov, V. V. Bulatov, and M. Ya. Kel'bert, "Intermediate asymptotic far field of internal waves in a layer of stratified liquid lying on a homogeneous layer," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 3 (1988).
4. V. F. Sannikov, "Near field of steady waves generated by a local source in a moving stratified liquid," in: Theoretical Studies of Wave Processes in the Ocean [in Russian], Marine Hydrophysics Institute, Sevastopol' (1983).
5. V. A. Borovikov, V. V. Bulatov, Yu. V. Vladimirov, and E. S. Levchenko, "Calculation of the field of internal gravity waves generated by a source at rest in a moving stratified liquid," Zh. Prikl. Mekh. Tekh. Fiz., No. 4 (1989).

6. I. S. Gradshteyn and I. M. Ryzhik (eds.), Tables of Integrals, Series, and Products, Academic Press, New York (1965).
7. Yu. Z. Miropol'skii, Dynamics of Internal Gravity Waves in the Ocean [in Russian], Gidrometeoizdat, Leningrad (1981).

OCCURRENCE OF THERMOCAPILLARY CONVECTION IN A CYLINDRICAL LAYER WITH DIFFERENT METHODS OF HEATING

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In the absence of body forces, perturbations of the equilibrium state of a nonuniformly heated fluid are governed by the intensity of the thermocapillary forces which arise as a result of the temperature dependence of surface tension. If the equilibrium temperature gradient is large enough, then a change in surface tension will lead to loss of stability of the equilibrium state – the occurrence of thermocapillary convection.

The studies [1-3] examined the conditions for the onset of convection in a fluid during heating of the solid or free boundary without allowance for the deformation of the free surface. Andreev et al. [4] studied the stability of the equilibrium of a fluid cylinder and cylindrical and plane layers heated by internal sources. The free surface was assumed to have been deformable in these cases. It was shown that allowance for the deformation of the boundary introduces a new factor which influences the stability of the equilibrium state. In this case, there is not only a decrease in stability, but there is a qualitative change in the neutral curve.

In the present investigation, we study the stability of a cylindrical layer with a deformable free surface in the case when the solid cylinder is also heated by internal sources. Formulas are obtained for the critical Marangoni numbers. It is shown that, as in [4], allowance for the deformation of the free boundary leads to discontinuities on the neutral curve. In the case of the heating of the solid surface, the curve of critical Marangoni numbers may have two points of discontinuity. Whether it does or not depends on the Weber number. Also, with heating by internal heat sources for azimuthal perturbations (m = 1), allowance for deformation of the free surface leads to an increase in stability.

1. We will examine a cylindrical layer of a viscous heat-conducting fluid bounded by a solid internal surface and free external surface. Gravitational forces are absent. We introduce a cylindrical coordinate system with the z axis directed along the generatrix of the cylinder. The equations of the solid and free boundaries $r = r_0$ and $r = r_1$, respectively. The change in surface tension as a function of temperature is described by the formula $\sigma = \sigma_0 - \varkappa(\theta - \theta_0)$.

Let the fluid contain permanent internal heat sources of intensity q, and let a constant temperature θ_1 be assigned for the solid boundary. Then the equilibrium state is written as

$$u = v = w = 0,\ p = \text{const},$$
$$\theta(r) = -\frac{q}{4\chi}\left[r^2 - r_1^2 + (r_1^2 - r_0^2)\frac{\ln(r/r_1)}{\ln(r_0/r_1)}\right] + \theta_1\frac{\ln(r/r_1)}{\ln(r_0/r_1)}, \tag{1.1}$$

where u, v, and w are components of the velocity vector; p is pressure; θ is temperature.

We choose the quantities r_1, r_1^2/ν, ν/r_1, $\rho\nu^2/r_1^2$, $\rho\nu^2/\varkappa r_1$ as the characteristic scales of length, time, velocity, pressure, and temperature (ν and χ are kinematic viscosity and diffusivity and ρ is density). After conversion to dimensionless form, the expression for temperature has the form

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